MATH4210: Financial Mathematics Tutorial 7

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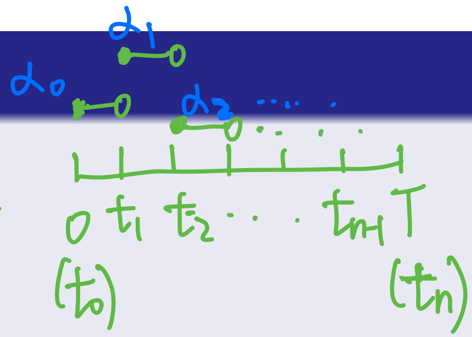
Content Review

Question



- *Let θ_t be a simple process.*

$$\int_0^T \theta_t dB_t = \sum_{i=0}^{n-1} \alpha_i (B_{t_{i+1}} - B_{t_i}) = \alpha_0 (B_{t_1} - 0) + \cdots + \alpha_{n-1}(B_{t_n} - B_{t_{n-1}}).$$

BM

- *For each $\theta_t \in \mathbb{H}^2([0, T])$, there exists a sequence of simple processes $(\theta_t^n)_{n \geq 1}$ such that*

$$\lim_{n \to \infty} \mathbb{E}\Big[\int_0^T (\theta_t - \theta_t^n)^2 dt \Big] = 0.$$

Content Review

Question

- *If $(\theta_t^n)_{n\geq 1}$ is a sequence of simple processes such that*

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_0^T (\theta_t - \theta_t^n)^2 dt\Big] = 0,$$

B_t

then the sequence of stochastic integrals $\int_0^T \theta_t^n dB_t$ has a unique limit in $\mathbb{L}^2(\Omega)$ as $n \to \infty$. And we define

$$\int_0^T \theta_t dB_t := \lim_{n\to\infty} \int_0^T \theta_t^n dB_t$$

B_t

as this limitation.

Content Review

Question

- *Let* $\theta_t \in \mathbb{H}^2([0,T])$, *then*

$$\mathbb{E}\Big[\int_0^T \theta_t dB_t\Big] = 0,\ \mathbb{E}\Big[\Big(\int_0^T \theta_t dB_t\Big)^2\Big] = \mathbb{E}\Big[\int_0^T \theta_t^2 dt\Big].$$

- *Let* H_t *be a adapted process and* $\Delta B_{k+1}^n := B_{t_{k+1}^n} - B_{t_k^n}$, *then* $\sum_{k=0}^{n-1} H_{t_k^n}((\Delta B_{k+1}^n)^2 - \Delta t) \to 0$, *in probability.*

BM

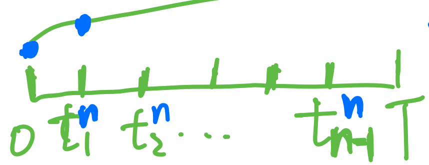
$n \to \infty$

$H_{t_k^n} = 1$

$\Delta B_1^n := B_{t_1^n} - B_{t_0^n}$

n

$\Delta t = \frac{T}{n}$



$\sum_{k=0}^{n-1}((\Delta B_{k+1}^n)^2 - \Delta t) \to 0$, in Prop

$\Rightarrow \sum_{k=0}^{n-1}(\Delta B_{k+1}^n)^2 - \sum_{k=0}^{n-1}\underbrace{\Delta t}_{\frac{T}{n}} \to 0$

$\Rightarrow \sum_{k=0}^{n-1}(\Delta B_{k+1}^n)^2 - T \to 0$, in Prob

Stochastic Integration

$\frac{T}{n}\cdot n$

$\overset{(n_j)}{\Longrightarrow} \sum_{k=0}^{n_j-1} (\Delta B_{k+1}^{n_j})^2 - T \to 0$, a.s

[$(X_n)_{n\geq 0}$; RV, $X_n \to X$, in Prob $\Rightarrow X_{n_j} \to X$, a.s]

Question

For fixed $T > 0$, prove that

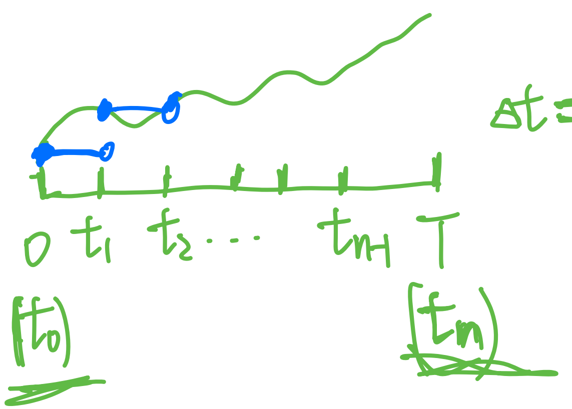
$$\int_0^T B_t dBt = \frac{1}{2}B_T^2 - \frac{1}{2}T$$

from sketch. $(B_t)_{t\geq 0}$ *is a standard Brownian motion.*

① find a seq of simple processes θ_t^n

② $\lim E[\int_0^T (B_t - \theta_t^n)^2 dt] = 0$

③ $\int_0^T B_t dB_t = \lim \int_0^T \theta_t^n dB_t$



$\Delta t = \frac{T}{n}$ ① $\theta_t^n := B_{t_k}$, $t \in [t_k, t_{k+1})$

Stochastic Integration

② $\lim_{n\to\infty} E\left[\int_0^T (B_t-\theta_t^n)^2\,dt\right]=0$

$$E\left[\int_0^T (B_t-\theta_t^n)^2\,dt\right]$$

$$=E\left[\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} (B_t-\underbrace{\theta_t^n}_{=B_{t_k}})^2\,dt\right]$$

$$=E\left[\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} (\underbrace{B_t-B_{t_k}}_{\sim N(0,\,t-t_k)})^2\,dt\right]$$

$$=\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} E(B_t-B_{t_k})^2\,dt$$

$$\left(\underbrace{Var(X)}_{=t-t_k}=E(X^2)-\underbrace{[E(X)]^2}_{=0}\right)$$

$$=\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} (t-t_k)\,dt \qquad t-t_k<\Delta t=\frac{T}{n}$$

$$\le \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \frac{T}{n}\,dt=\sum_{k=0}^{n-1}\frac{T}{n}\cdot\underbrace{(t_{k+1}-t_k)}_{=\Delta t=\frac{T}{n}}$$

③ $\int_0^T B_t\,dB_t=\lim_{n\to\infty}\int_0^T \theta_t^n\,dB_t$

$$\int_0^T \theta_t^n\,dB_t=\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \underbrace{\theta_t^n}_{=B_{t_k}}\,dB_t$$

$$=\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} B_{t_k}\,dB_t$$

$$=\sum_{k=0}^{n-1} B_{t_k}(B_{t_{k+1}}-B_{t_k})$$

$$=\sum_{k=0}^{n-1} B_{t_k}B_{t_{k+1}}-B_{t_k}^2$$

$$=\sum_{k=0}^{n-1} -\frac{1}{2}\left(-2B_{t_k}B_{t_{k+1}}+2B_{t_k}^2\right)$$

$$=\sum_{k=0}^{n-1} -\frac{1}{2}\left[(B_{t_k}-B_{t_{k+1}})^2-B_{t_{k+1}}^2+B_{t_k}^2\right]$$

$$=\sum_{k=0}^{n-1} -\frac{1}{2}\underbrace{\left[(B_{t_k}-B_{t_{k+1}})^2\right]}_{\Delta B_{t_k}^2}+\sum_{k=0}^{n-1}\left(\frac{1}{2}\right)\left(B_{t_{k+1}}^2-B_{t_k}^2\right)$$

$$\left(B_{t_n}^2-\cancel{B_{t_{n-1}}^2}+\cancel{B_{t_{n-1}}^2}-\cancel{B_{t_{n-2}}^2}+\cancel{\cdots}+B_{t_1}^2-B_{t_0}^2\right)$$

$$=\sum_{k=0}^{n-1}-\frac{1}{2}\Delta B_{t_k}^2+\frac{1}{2}B_T^2$$

$\left(\sum_{k=0}^{n-1}\Delta B_{t_k}^2\to T \text{ a.s.}\right)$

Ito Formula $= \frac{T}{n} \cdot \frac{T}{n} \cdot n = \frac{T^2}{n} \to 0$

$\xrightarrow{n\to\infty} -\frac{1}{2}T + \frac{1}{2}B_T^2$

$B_{t_n}^2 - B_{t_0}^2$, $B_{t_n}^2 = B_T^2$, $B_{t_0}^2 = 0$

(a) $Y_t = f(B_t)$

$dY_t = f'(B_t)\,dB_t + \frac{f''(B_t)}{2}(dB_t)^2$, $(dB_t)^2 = dt$ $\quad \left(dB_t \sim O(t^{\frac{1}{2}}),\ dt \sim O(t^1) \right)$

Question

Consider a standard Brownian motion $(B_t)_{t\geq 0}$. Let $T > 0$, compute

(a) $\int_0^T B_t dBt$ $\rightsquigarrow \int_0^T d\left(\frac{B_t^2}{2}\right)$

(b) $\int_0^T \exp(B_t - \frac{1}{2}t)dt$

using Ito formula.

$f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$

$\Rightarrow B_T^2 - B_0^2 = 2\int_0^T B_t dB_t + T - 0$ ($0 \to T$, $B_0 = 0$)

$df(B_t) = f'(B_t)dB_t + \frac{f''(B_t)}{2}dt \Rightarrow \int_0^T B_t dB_t = \frac{B_T^2}{2} - \frac{T}{2}$

$\Rightarrow dB_t^2 = 2 \cdot B_t \cdot dB_t + dt$

$dB_t \cdot dt = o(t^{1.5})$

$o(t^{\frac{1}{2}}) \quad o(t^1)$

(b) $Y_t := f(t, B_t)$

$o(t^1)$

$$dY_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dB_t + \cancel{\frac{\partial^2 f}{\partial x \partial t} dB_t dt} + \frac{\partial^2 f}{\partial x^2} \cdot \frac{1}{2} \underbrace{(dB_t)^2}_{dt}$$

$$= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dB_t + \frac{\partial^2 f}{\partial x^2} \cdot \frac{1}{2} dt = \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right) dt + \frac{\partial f}{\partial x} dB_t$$

$f(t,x) = \exp(x - \frac{1}{2}t)$; $\frac{\partial f}{\partial t} = e^{x - \frac{1}{2}t} \cdot (\frac{-1}{2})$; $\frac{\partial f}{\partial x} = e^{x - \frac{1}{2}t}$; $\frac{\partial^2 f}{\partial x^2} = e^{x - \frac{1}{2}t}$

$$df(t, B_t) = \underbrace{\left(e^{B_t - \frac{1}{2}t} \cdot (\frac{-1}{2}) + \frac{1}{2} \cdot e^{B_t - \frac{1}{2}t} \right)}_{= 0} dt + e^{B_t - \frac{1}{2}t} dB_t$$

$$\overset{\int_0^T}{\Longrightarrow} f(T, B_T) - f(0, B_0) = \int_0^T e^{B_t - \frac{1}{2}t} dB_t$$

$$e^{B_T - \frac{1}{2}T} - \underbrace{e^0}_{=1} = \int_0^T e^{B_t - \frac{1}{2}t} dB_t \Longrightarrow \int_0^T e^{B_t - \frac{1}{2}t} dB_t = e^{B_T - \frac{1}{2}T} - 1$$